Solution of ordinary differential equations and Volterra integral equation of first and second kind with bulge and logarithmic functions using Laplace transform

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ABSTRACT

A large class of complications of mathematical physics, applied mathematics and engineering are formulated in the form of differential equations, beside with few additional conditions. This paper comprises of an ordinary differential equation (O.D.E) and Volterra Integral equation (V.I.E) with bulge and logarithmic functions. We will use Laplace transform, Inverse Laplace transform and convolution theorem where it will be needed to find the precise solution of O.D.Es and V.I.Es. Also, we will compare it with the numerical solution using Euler's method and Simpson's quadrature rule and lastly we will represent it with the help of graph.

1. Introduction

Integral equations are significant in numerous applications. Problems in which integral equations are faced include radiative energy transfer and the oscillation of a string, membrane, or axle. Oscillation problems may also be solved as differential equations. Song and Kim (2014) discovered the solution of Volterra integral equation of the second kind by using the Elzaki transform. Mirzaee (2012) introduced a numerical method for solving linear Volterra integral equations of the second kind based on the adaptive Simpson's quadrature method. They also derived a simple and efficient matrix formulation using Chebyshev polynomials as trial functions.

Many researchers have established the different numerical methods to solve the Volterra integral equation by using different polynomials (Saran et al., 2000). O.D.Es occur in many scientific disciplines, e.g., in chemistry, physics, biology, and economic. Differential equation explains the changes in population, movement of heat, vibration of spring, how radioactive material decays and many other things. These are the ordinary ways to describe different things in this universe.

To find the analytical solution of the differential equation we can use integration method but unluckily in different practical applications like engineering and science, we have to find their numerical solutions rather than analytical solutions.

In this paper we have discussed solution of ordinary differential equations and Volterra integral equation with different functions (like Bulge and Logarithmic function) by using Laplace transform. Examples are also there to show the efficiency of these methods. We have used here Laplace transform, inverse Laplace transform and convolution theorem to find the exact solution of O.D.Es with bulge and logarithmic function. Euler's method is used here to find the numerical solution of O.D.Es. In the end we will compare the results of numerical and exact solutions using graphs.

2. Preliminaries

We start our study by giving the definitions of O.D.E, V.I.E, Laplace transform, convolution theorem and Simpson's quadrature rule, which can be used in this study.

2.1. The Laplace transform

Let f(t) be continuous function of t defined over the interval $[0, \infty)$, then the Laplace transform (Henry et al., 2004) of f(t) is a function F(s) of another variable s defined by:

$$F(s) = \int_0^\infty e^{-st} \, f(t)dt = \lim_{T \to \infty} \int_0^T e^{-st} \, f(t)dt$$

2.2. The Volterra integral equations

Integral equations given in (Lomen and Mark, 1988) are a special kind of integral equations. One type has the form

$$y(t) = f(t) + \int_0^t k(t - \tau)y(\tau)d(\tau)$$

where f and k are known and y is to be determined.

2.3. The convolution theorem

The convolution of two functions f(t) and g(t) denoted f(t)∗g(t), is given by $f(t) * g(t) = \int_0^t f(t)g(t - \tau)d\tau$, whenever the integral is defined. For this paper, we study the case that f(t) is a bulge function which is given by f(t) = $e^{-\frac{(t-l)^2}{2}}$ where l is a positive constant (Lomen and Mark, 1988).

2.4. Simpson's quadrature rule

The Simpson's quadrature rule can be used for the numerical solutions of the Volterra integral equation of the first kind. If N is even, then Simpson's quadrature rule may be applied to each subinterval $[x_i, \ x_{2i+1}]$ individually yields the approximation

$$S(h) = \frac{h}{3}[f(a) + 4f(a + h) + 2f(a + 2h) + 4f(a + 3h) + \cdots + 2f(b - 2h) + 4f(b - h) + f(b)]$$

for the complete interval (Phillips and Taylor, 1973; Mirzaee, 2012). The error of S (h) is

$$E \, S \, (h) = \int f(x) \, dx\text{-}s \, (h) = \frac{L^5}{90} \sum_{i=0}^{\frac{N}{2}} f^4(\xi i), \quad \xi i = [x_i, \ x_{2i+1}]$$

3. Solution of O.D.E using Laplace transform

3.1. Lemma

The L.T. of the Bulge function $e^{-\frac{(t-l)^2}{2}}$ is expressed by (Harsa and Pothat, 2015)

$$L\left\{e^{-\frac{(t-l)^2}{2}}\right\} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right] \tag{1}$$

Proof: The Taylor series expression e^x is of the form:

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

therefore by substituting with $x = -\frac{(t-l)^2}{2}$, we obtain

$$e^{-\frac{(t-l)^2}{2}} = e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}}lt + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^2}{2}\right)t^2 + e^{-\frac{l^2}{2}}\left(-\frac{1}{2} + \frac{l^3}{2}\right)t^3 + O(t^4).$$

By taking L.T. of above equation and neglecting higher order terms and using the fact that the L.T. is linear, we derived

$$L\left\{e^{-\frac{(t-l)^2}{2}}\right\} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right]. \tag{2}$$

3.2. Theorem

The linear O.D.E with the Bulge function

$$\frac{dy}{dt} - 1 + y = e^{-\frac{(t-l)^2}{2}} \tag{3}$$

and solution can be written as:

$$y(t) = e^{-t} e^{-\frac{l^2}{2}}\left[1 + lt + \frac{t^2}{2}(-1 + l^2) + \frac{t^3 l}{6}(-3 + l^2) + e^{\frac{l^2}{2}}\right] \tag{4}$$

Proof: Eq. 3 can be written as:

$$y' - 1 + y = e^{-\frac{(t-l)^2}{2}}$$

By taking L.T. of above equation

$$sY(s) - y(0) + Y(s) - \frac{1}{s} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right] \tag{5}$$

By putting initial condition y (0) =0

$$(s + 1)Y(s) - \frac{1}{s} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right]$$

$$Y(s) = \frac{e^{-\frac{l^2}{2}}}{(s+1)}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4} + \frac{e^{\frac{l^2}{2}}}{s}\right]. \tag{6}$$

Now by taking ILT and using convolution theorem

$$y(t) = e^{-t}e^{-\frac{l^2}{2}}\left[1 + lt + \frac{t^2}{2}(-1 + l^2) + \frac{t^3 l}{6}(-3 + l^2) + e^{\frac{l^2}{2}}\right]. \tag{7}$$

3.3. Comparison of approximate and exact solution

In above Eq. 3 with initial condition y (0) =0, l = 2, h = 0.1, Eq. 7 is its exact solution and by applying Euler's method we will obtain its numerical solution. Table 1 and Fig. 1 show the comparative analysis.

3.4. Theorem

The linear O.D.E with the Bulge function

$$\frac{dy}{dt} + y = e^{-\frac{(t-l)^2}{2}} \tag{8}$$

and solution can be written as:

$$y(t) = e^{-t} e^{-\frac{l^2}{2}}\left[1 + lt + \frac{t^2}{2}(-1 + l^2) + \frac{t^3 l}{6}(-3 + l^2) + 2e^{\frac{l^2}{2}}\right] \tag{9}$$

Table 1: Approximate and exact and solution

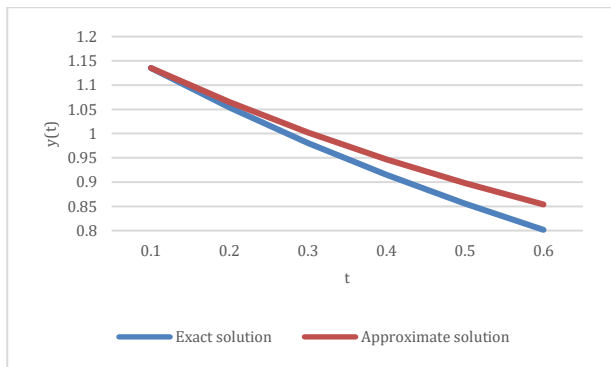| t	Exact solution	Approximate solution	Error
0.1	1.1353	1.1353	0
0.2	1.0536	1.0647	0.0111
0.3	0.9805	1.0022	0.0217
0.4	0.9148	0.9469	0.0321
0.5	0.8554	0.8977	0.0423
0.6	0.8015	0.8538	0.0523

Fig. 1: Approximate and exact and solution for $l = 2, h = 0.1$

Proof: Eq. 8 can be written as:

$$y' + y = e^{-\frac{(t-l)^2}{2}}$$

by taking L.T. of above equation

$$sY(s) - y(0) + Y(s) = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right] \qquad (10)$$

by putting initial condition $y(0) = 2$

$$(s + 1)Y(s) = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right] + 2$$

$$Y(s) = \frac{e^{-\frac{l^2}{2}}}{(s+1)}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right] + \frac{2}{s+1} \qquad (11)$$

now by taking ILT and using convolution theorem

$$y(t) = e^{-t}e^{-\frac{l^2}{2}}\left[1 + lt + \frac{t^2}{2}(-1+l^2) + \frac{t^3l}{6}(-3+l^2) + 2e^{\frac{l^2}{2}}\right]. \qquad (12)$$

3.5. Comparison of approximate and exact solution

In above Eq. 8 with initial condition $y(0) = 2$ and by taking $l = 2, h = 1$ in Euler's method we will obtain its numerical solution. Table 2 and Fig. 2 show the comparative analysis.

Table2: Approximate and exact and solution

t	Exact solution	Approximate solution	Error
1	0.9764	0.9394	0.037
2	0.5210	0.5689	0.0479
3	0.2983	0.3292	0.0309
4	0.1713	0.1677	0.0036
5	0.0957	0.0872	0.0085
6	0.0516	0.0465	0.0051
7	0.0269	0.0242	0.0027
8	0.0135	0.0122	0.0003

4. The solution of Volterra integral equation of the first kind with bulge function using the Laplace transform

4.1. Theorem

The solution of the VIE of the first kind

$$\int_0^t y(t-\eta)e^{a\eta}\,d\eta = e^{-\frac{(t-l)^2}{2}} \qquad (13)$$

can be expressed as

$$y(t) = e^{-\frac{l^2}{2}}[6 - 6a + 3(a - al^2 - 3l + l^3)t^2 + 6(-1 + l^2 - al)t + l(6 + 3at^3 - l^2at^3)].$$

Proof: By taking L.T to the Eq. 13,

$$L\left\{\int_0^t y(t-\eta)e^{a\eta}\,d\eta\right\} = L\left\{e^{-\frac{(t-l)^2}{2}}\right\}.$$
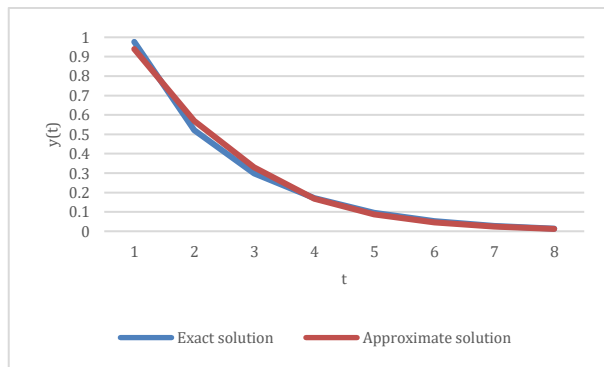

Fig. 2: Approximate and exact and solution for $l = 2, h = 1$

By applying the convolution theorem, it will become

$$L\{y(t) * e^t\} = L\left\{e^{-\frac{(t-l)^2}{2}}\right\} \qquad (14)$$

again by using convolution theorem and lemma (3.1)

$$L\{y(t)\}\,L\{e^t\} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right].$$

or

$$L\{y(t)\}\left(\frac{1}{s-1}\right) = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right]. \qquad (15)$$

Then after interpretation of Eq. 15 it will become

$$L\{y(t)\} = e^{-\frac{l^2}{2}}\left[1 + \frac{-a+l}{s} + \frac{-1+l^2-al}{s^2} + \frac{a-al^2-3l+l^3}{s^3} + \frac{al(3-l^2)}{s^4}\right]. \qquad (16)$$

Now by taking ILT of Eq. 16 we will obtain

$$y(t) = e^{-\frac{l^2}{2}}[6 - 6a + 3(a - al^2 - 3l + l^3)t^2 + 6(-1 + l^2 - al)t + l(6 + 3at^3 - l^2at^3)] \qquad (17)$$

4.2. Theorem

We can express the solution of the V.I.E of the first kind

$$\int_0^t e^{t-x}g(x)\,dx = e^{-\frac{(t-l)^2}{2}} \qquad (18)$$

and solution can be written as:

$$g(t) = \frac{e^{-\frac{l^2}{2}}}{6}[6t + 6(-1 + l) + 6t(-1 - l + l^2) + 3t^2(1 - l^2 - 3l + l^3) + t^3(3l - l^3)]$$

Proof: Applying L.T. to the Eq. 18,

$L\left\{\int_0^t e^{t-x} g(x)\, dx\right\} = L\left\{e^{-\frac{(t-l)^2}{2}}\right\}$

by using the convolution theorem, it will become

$$L\{g(t)*e^t\}= L\left\{e^{-\frac{(t-l)^2}{2}}\right\} \qquad (19)$$

again by using convolution theorem and lemma 3.1 we get

$$L\{g(t)\}\, L\{e^t\} = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right]$$

or

$$L\{g(t)\}\left(\frac{1}{s-1}\right)=e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1+l^2}{s^3} + \frac{l(s^2-3+l^2)}{s^4}\right]. \qquad (20)$$

Then after simplification of Eq. 20 will become

$$L\{g(t)\} = e^{-\frac{l^2}{2}}\left[1 + \frac{-1+l}{s} + \frac{-1-l+l^2}{s^2} + \frac{1-l^2-3l+l^3}{s^3} + \frac{3l-l^3}{s^4}\right].$$

Now by taking ILT to above equation to obtain

$$g\ (t)\ =\ e^{-\frac{l^2}{2}}\left[t + (-1 + l) + t(-1 - l + l^2) + \frac{t^2}{2}(1 - l^2 - 3l + l^3) + \frac{t^3}{6}(3l - l^3)\right].$$

or

$$g(t)=\frac{e^{-\frac{l^2}{2}}}{6}\left[6t + 6(-1 + l) + 6t(-1 - l + l^2) + 3t^2(1 - l^2 - 3l + l^3) + t^3(3l - l^3)\right]. \qquad (21)$$

5. Solution of ordinary differential equation with a logarithmic function

5.1. Lemma

The Laplace transform of the logarithmic function $\ln t$ is expressed by:

$$L\{\ln t\} = \left[-\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4}\right] \qquad (22)$$

Proof:

$\ln t = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(t-1)^n}{n}$

$\ln t = t - 1 + \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \frac{(t-1)^4}{4} + \cdots \ldots \ldots \ldots ..$

$0 \le t \le 2$

by neglecting higher order terms

$$\ln t = -\frac{11}{6} + 3t - \frac{3t^2}{2} + \frac{t^3}{3} \qquad (23)$$

now by taking Laplace transform of Eq. 23

$$L\{lnt\} = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4}.$$

5.2. Theorem

The linear O.D.E with the logarithmic function

$$\frac{dy}{dt} - 1 + y = \ln t \qquad (24)$$

with initial condition $y(0) = 0$. and solution can be written as:

$$y(t) = e^{-t}\left(-\frac{5}{6} + 3t - \frac{3t^2}{2} + \frac{t^3}{3}\right) \qquad (25)$$

Proof: Eq. 24 can be written as:

$$y' - 1 + y = \ln t$$

By taking L.T of above equation and by using lemma 5.1

$$sY(s) - y(0) + Y(s) - \frac{1}{s}=-\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} \qquad (26)$$

by applying initial condition $y(0) = 0$

$$(s + 1)Y(s) - \frac{1}{s} = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4}$$

or

$$y(s) = \frac{1}{(s+1)}\left(-\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} + \frac{1}{s}\right) \qquad (27)$$

Now by taking ILT and using convolution theorem

$$y(t) = e^{-t}\left(-\frac{5}{6} + 3t - \frac{3t^2}{2} + \frac{t^3}{3}\right). \qquad (28)$$

5.3. Comparative analysis of approximate and exact solution

In above equation (24) with initial condition $y(0) = 0$ and by taking $h = 1$ we solve it by Euler's method to get numerical solution. Table 3 and Fig. 3 show the comparative analysis.

Table3: Approximate and exact and solution

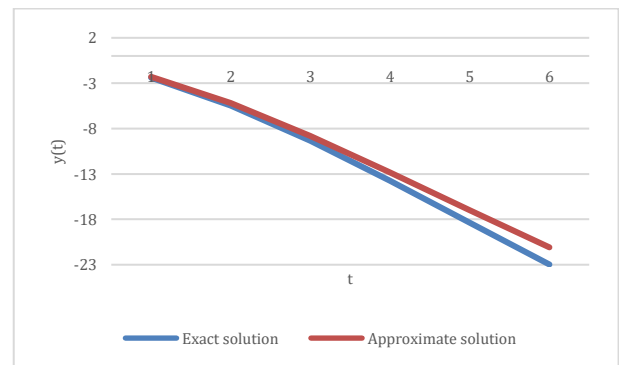| t	Exact solution	Approximate solution	Error
1	-2.3964	-2.2964	0
2	-5.4542	-5.1901	-0.2641
3	-9.3547	-8.8083	-0.5464
4	-13.7594	-12.8310	-0.9284
5	-18.3852	-17.0002	-1.385
6	-22.9746	-21.0868	-1.8878

Fig. 3: Approximate and exact and solution of example for $h = 1$

5.4. Theorem

The linear O.D.E with the logarithmic function

$$\frac{dy}{dt} + y = \ln t \qquad (29)$$

with initial condition y (0) =2. And solution can be written as:

$$y(t) = e^{-t}\left(\frac{1}{6} + 3t - \frac{3t^2}{2} + \frac{t^3}{3}\right) \qquad (30)$$

Proof: Eq. 29 can be written as $y' + y = \ln t$ by taking L.T of above equation and using lemma 5.1

$$sY(s) - y(0) + Y(s) = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} \qquad (31)$$

by putting initial condition y (0) =2

$$(s+1)Y(s) = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} + 2$$
$$Y(s) = \frac{1}{(s+1)}\left(-\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} + \frac{1}{s} + 2\right) \qquad (32)$$

now by taking ILT and using convolution theorem

$$y(t) = e^{-t}\left(-\frac{11}{6} + 3t - 3t^2 + \frac{t^3}{3} + 2\right) \text{ or}$$

$$y(t) = e^{-t}\left(\frac{1}{6} + 3t - \frac{3t^2}{2} + \frac{t^3}{3}\right) \qquad (33)$$

5.5. Comparative analysis of approximate and exact solution

In above Eq. 29 with initial condition y (0) =2 and by taking h = 1.02 in Euler's method we will get numerical solution. Table 4 and Fig. 4 show the comparative analysis.

Table 4: Approximate and exact and solution

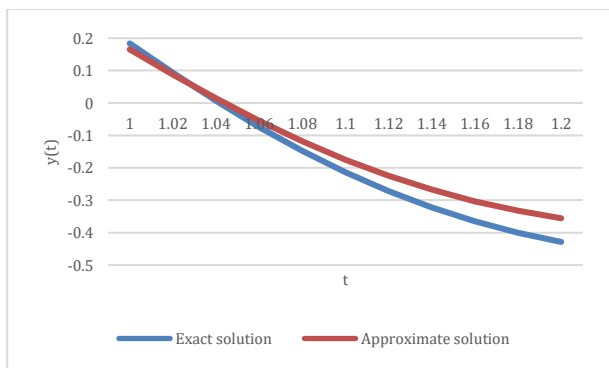| t	Exact solution	Approximate solution	Error
1	0.1839	0.1655	0.0184
1.02	0.0933	0.0881	0.0052
1.04	0.0068	0.0141	0.0073
1.06	-0.0739	-0.0551	0.0188
1.08	-0.1476	-0.1183	0.0293
1.1	-0.2138	-0.1748	0.039
1.12	-0.2722	-0.2245	0.0477
1.14	-0.3226	-0.2673	0.0553
1.16	-0.3652	-0.3032	0.062
1.18	-0.4004	-0.3325	0.0679
1.2	-0.4286	-0.3556	0.073s

Fig. 4: Approximate and exact and solution for h = 1.02

6. Solution of Volterra integral equation of the second kind with a logarithmic function

6.1. Theorem

We can express the solution of the V.I.E of the second kind

$$u(t) - \int_0^t \sin(t-y)\, u(y)\, dy = \ln t. \qquad (34)$$

is expressed by:

$$u(t) = -\frac{11}{6} + 3t - \frac{29}{12}t^2 + \frac{5}{6}t^3 - \frac{t^4}{8} + \frac{t^5}{60}.$$

Proof: First we will apply the L.T to the Eq. 34,

$$L\{u(t)\} - L\left\{\int_0^t \sin(t-y)\, u(y)\, dy\right\} = L\{\ln t\};$$

by using the convolution theorem, it will become

$$L\{u(t)\} - L\{y(t) * \sin t\} = L\{\ln t\}. \qquad (35)$$

Again by applying convolution theorem and lemma 5.1, Eq. 35 will become

$$L\{u(t)\} - L\{u(t)\}\left(\frac{1}{s^2+1}\right) = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4}$$

or

$$L\{u(t)\}\left[1 - \frac{1}{s^2+1}\right] = -\frac{11}{6s} + \frac{3}{s^2} - \frac{3}{s^3} + \frac{2}{s^4} \qquad (36)$$

then after simplification of Eq. 36 will become

$$L\{u(t)\} = \left[\frac{-11}{6s} + \frac{3}{s^2} - \frac{29}{6s^3} + \frac{5}{s^4} - \frac{3}{s^5} + \frac{2}{s^6}\right]; \qquad (37)$$

now by taking ILT to Eq. 37 to obtain

$$u(t) = -\frac{11}{6} + 3t - \frac{29}{12}t^2 + \frac{5}{6}t^3 - \frac{t^4}{8} + \frac{t^5}{60}. \qquad (38)$$

6.2. Comparative analysis of approximate and exact solution

We will obtain exact solution of Eq. 34 by using Simpson's quadrature formula taking h=0.01. Table 5 and Fig. 5 show the comparative analysis.

Table 5: Approximate and exact and solution

t	Exact solution	Approximate solution	Error
0	-1.8333	-1.8333	0
0.01	-1.8037	-1.8036	-0.0001
0.02	-1.7749	-1.7745	-0.0005
0.03	-1.7468	-1.7460	-0.0008
0.04	-1.7195	-1.7181	-0.0014
0.05	-1.6930	-1.6908	-0.0022
0.06	-1.6673	-1.6641	-0.0032
0.07	-1.6422	-1.6379	-0.0043
0.08	-1.6180	-1.6124	-0.0056

7. Conclusion

In this work, we have studied the V.I. equations of the first and second kind and O.D.E with the bulge function $e^{-\frac{(t-l)^2}{2}}$ and logarithmic function. To solve the numerical solution of the V.I.E we have used here Simpson's quadrature rule and to solve the O.D.E we have used Euler's method. We have found the exact solution by applying the L.T. There is also the comparison of exact and approximate solutions through graphical representation.
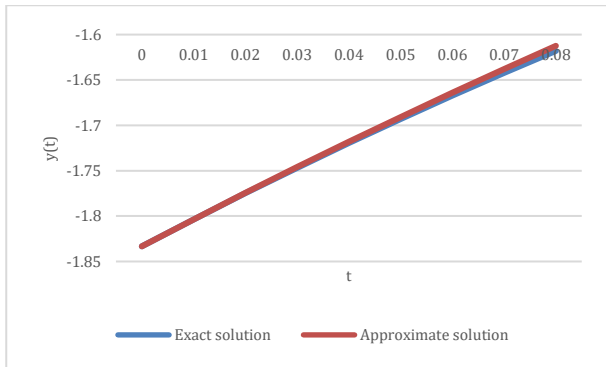
Fig. 5: Approximate and exact and solution h = 0.01

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